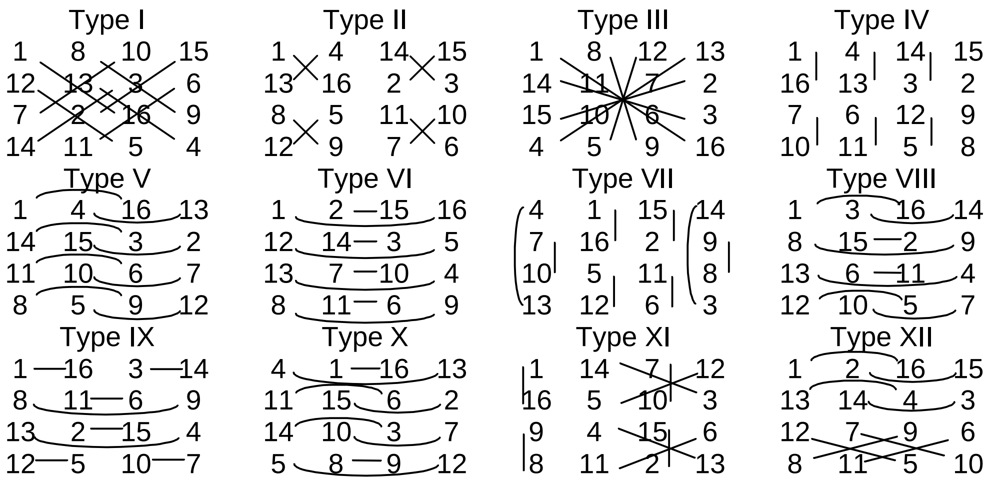
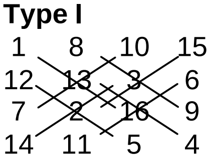
12 TYPES OF ORDER 4 MAGIC SQUARES

H. E. Dudeney categorised all order 4 magic squares into 12 types according to how the pairs of numbers that add up to 17 are arranged:

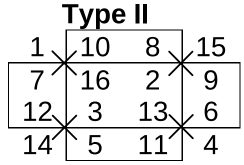


For any order 4 magic square, exchanging rows 2 and 3, followed by exchanging columns 2 and 3 results in another magic square. Let us call this procedure a ‘2-3 swap’.
A 2-3 swap will transform a type III or VI magic square into another of the same type.
A 2-3 swap will transform a type I magic square into a type II (and vice versa), a type IV into a type V (and vice versa), a type VII into a type X (and vice versa), a type VIII into a type IX (and vice versa), and a type XI into a type XII (and vice versa). For example,

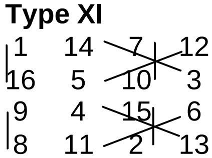
Type I



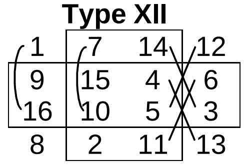
Type II



Type XI

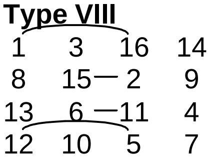


Type XII

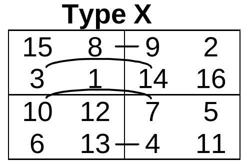


For any order 4 magic square, exchanging rows 1 and 2, exchanging rows 3 and 4, and then exchanging columns 1 and 2, and finally exchanging columns 3 and 4 also results in another magic square. Let us call this procedure a ‘1-2, 3-4 swap’.
A 1-2, 3-4 swap will transform a type I, II, III, IV, V, VI, XI or XII magic square into another of the same type.
A 1-2, 3-4 swap will transform a type VII magic square into a type IX (and vice versa), and a type VIII into a type X (and vice versa). For example,

Type VIII



Type X



For any order 4 magic square, subtracting each number from 17 results in another magic square of the same type. The resulting square is called the ‘complement’ of the original square. Magic squares of type III and VI are self-complementary, i.e. the complement of each square is itself.

Each magic square generates 31 others by reflection, rotation, 2-3 swap & 1-2,3-4 swap:

		rotate 90° clockwise	**rotate 180°**	**rotate 90° anticlockwise**
	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16	13 9 5 1 14 10 6 2 15 11 7 3 16 12 8 4	16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1	4 8 12 16 3 7 11 15 2 6 10 14 1 5 9 13
reflect horizontally	13 14 15 16 9 10 11 12 5 6 7 8 1 2 3 4	1 5 9 13 2 6 10 14 3 7 11 15 4 8 12 16	4 3 2 1 8 7 6 5 12 11 10 9 16 15 14 13	16 12 8 4 15 11 7 3 14 10 6 2 13 9 5 1
2-3 swap	1 3 2 4 9 11 10 12 5 7 6 8 13 15 14 16	13 5 9 1 15 7 11 3 14 6 10 2 16 8 12 4	16 14 15 13 8 6 7 5 12 10 11 9 4 2 3 1	4 12 8 16 2 10 6 14 3 11 7 15 1 9 5 13
reflect horizontally, 2-3 swap	13 15 14 16 5 7 6 8 9 11 10 12 1 3 2 4	1 9 5 13 3 11 7 15 2 10 6 14 4 12 8 16	4 2 3 1 12 10 11 9 8 6 7 5 16 14 15 13	16 8 12 4 14 6 10 2 15 7 11 3 13 5 9 1
1-2, 3-4 swap	6 5 8 7 2 1 4 3 14 13 16 15 10 9 12 11	10 14 2 6 9 13 1 5 12 16 4 8 11 15 3 7	11 12 9 10 15 16 13 14 3 4 1 2 7 8 5 6	7 3 15 11 8 4 16 12 5 1 13 9 6 2 14 10
1-2, 3-4 swap, reflect horizontally	10 9 12 11 14 13 16 15 2 1 4 3 6 5 8 7	6 2 14 10 5 1 13 9 8 4 16 12 7 3 15 11	7 8 5 6 3 4 1 2 15 16 13 14 11 12 9 10	11 15 3 7 12 16 4 8 9 13 1 5 10 14 2 6
1-2, 3-4 swap, 2-3 swap	6 8 5 7 14 16 13 15 2 4 1 3 10 12 9 11	10 2 14 6 12 4 16 8 9 1 13 5 11 3 15 7	11 9 12 10 3 1 4 2 15 13 16 14 7 5 8 6	7 15 3 11 5 13 1 9 8 16 4 12 6 14 2 10
1-2, 3-4 swap, reflect horizontally, 2-3 swap	10 12 9 11 2 4 1 3 14 16 13 15 6 8 5 7	6 14 2 10 8 16 4 12 5 13 1 9 7 15 3 11	7 5 8 6 3 1 4 2 15 13 16 14 11 9 12 10	11 3 15 7 9 1 13 5 12 4 16 8 10 2 14 6

Type	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	**Total**
No. of magic squares not counting reflection and rotation	48	48	48	96	96	304	56	56	56	56	8	8	**880**

Type	I & II	III	IV & V	VI	VII, VIII, IX & X	XI & XII	**Total**
No. not counting reflection, rotation, 2-3 swap & 1-2, 3-4 swap	24	12	48	76	56	4	**220**

Type II

Reflecting the left or right half along a vertical mirror, or the top or bottom half along a horizontal mirror results in another type II magic square. For example,

1	4	14	15
13	16	2	3
8	5	11	10
12	9	7	6

1	4	14	15
13	16	2	3
12	9	7	6
8	5	11	10

Type III

Reflecting the middle 2 rows along a horizontal mirror results in another type III magic square.
Reflecting the left and right halves along a vertical mirror also results in another type III magic square. For example,

1	8	12	13
14	11	7	2
15	10	6	3
4	5	9	16

1	8	12	13
15	10	6	3
14	11	7	2
4	5	9	16

8	1	13	12
11	14	2	7
10	15	3	6
5	4	16	9

Consider a type III magic square:

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

$a + b + c + d = 34$
$c + d + m + n = 34$
$\therefore a + b = m + n$
Similarly $g + h = k + l$
$a + b + e + f + i + j + m + n = 34 + 34 = 68$
$e + f + g + h + i + j + k + l = 34 + 34 = 68$
$\therefore a + b + m + n = g + h + k + l$
But $a + b = m + n$ and $g + h = k + l$
$\therefore a + b = m + n = g + h = k + l$
Similarly $c + d = e + f = i + j = o + p$
$a + e = j + n = k + o = d + h$
$b + f = c + g = i + m = l + p$

Therefore exchanging 2 diagonally opposite 2 × 2 squares also results in another type III magic square. For example,

1	8	12	13
14	11	7	2
15	10	6	3
4	5	9	16

1	8	15	10
14	11	4	5
12	13	6	3
7	2	9	16

Exchanging blocks of 2 cells as shown below also results in another type III magic square. For example,

1	8	12	13
14	11	7	2
15	10	6	3
4	5	9	16

1	8	14	11
12	13	7	2
15	10	4	5
6	3	9	16

Type IV

Rotating the upper or lower 2 × 2 squares 180° results in another type IV magic square. Reflecting any 2 diagonally opposite 2 × 2 squares along a vertical mirror also results in another type IV magic square. For example,

1	4	14	15
16	13	3	2
7	6	12	9
10	11	5	8

1	4	14	15
16	13	3	2
11	10	8	5
6	7	9	12

1	4	15	14
16	13	2	3
6	7	12	9
11	10	5	8

Consider a type IV magic square:

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

$b + c + f + g = 34$
$e + f + g + h = 34$
$\therefore b + c = e + h$
Similarly $j + k = m + p$
$a + b + c + d + e + f + g + h = 34 + 34 = 68$
$a + f + k + p + d + g + j + m = 34 + 34 = 68$
$\therefore b + c + e + h = k + p + j + m$
But $b + c = e + h$ and $j + k = m + p$
$\therefore b + c = e + h = j + k = m + p$
Similarly $a + d = f + g = i + l = n + o$
Also $a + f + k + p = 34$
$a + e + l + p = 34$
$\therefore f + k = e + l$
Similarly $c + n = d + m$
$c + d + g + h + i + j + m + n = 4 \times 17 = 68$
$e + f + g + h + i + j + k + l = 4 \times 17 = 68$
$\therefore c + d + m + n = e + f + k + l$
But $f + k = e + l$ and $c + n = d + m$
$\therefore f + k = e + l = c + n = d + m$

Therefore exchanging the central top and bottom 2 × 2 squares results in another type IV magic square. For example,

1	4	14	15
16	13	3	2
7	6	12	9
10	11	5	8

1	6	12	15
16	11	5	2
7	4	14	9
10	13	3	8

Consider a type IV magic square:

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

$b + d + f + h = 34$
$e + f + g + h = 34$
$\therefore b + d = e + g$
Similarly $i + k = n + p$
$a + f + k + p = 34$
$k + l + o + p = 34$
$\therefore a + f = l + o$
Similarly $c + h = j + m$

$a + b + c + d + e + f + g + h = 34 + 34 = 68$
$a + f + c + h + i + k + n + p = 34 + 34 = 68$
$\therefore b + d + e + g = i + k + n + p$
But $b + d = e + g$ and $i + k = n + p$
$\therefore b + d = e + g = i + k = n + p$
Similarly $a + c = f + h = j + l = m + o$
$4(b + d + j + l) = (b + d) + (e + g) + (i + k) + (n + p) + (a + c) + (f + h) + (j + l) + (m + o) = 136$
$\therefore b + d + j + l = 136 \div 4 = 34$
$b + f + l + p = 17 + 17 = 34$
$\therefore d + j = f + p$

Therefore reflecting the 2 alternate columns along a horizontal mirror results in another type IV magic square. For example,

1	4	14	15
16	13	3	2
7	6	12	9
10	11	5	8

1	11	14	8
16	6	3	9
7	13	12	2
10	4	5	15

Type VI

Reflecting both the left and right middle 2 cells results in another type VI magic square. Exchanging the top and bottom middle 2 cells also results in another type VI magic square. For example,

1	2	15	16
12	14	3	5
13	7	10	4
8	11	6	9

1	2	15	16
13	14	3	4
12	7	10	5
8	11	6	9

1	11	6	16
12	14	3	5
13	7	10	4
8	2	15	9

Type VIII

Reflecting the left and right middle 2 cells results in another type VIII magic square. For example,

1	3	16	14
8	15	2	9
13	6	11	4
12	10	5	7

1	3	16	14
13	15	2	4
8	6	11	9
12	10	5	7

40 Fundamental Magic Squares of Order 4

The magic squares listed below are fundamental ones from which all other magic squares of order 4 can be generated by reflection, rotation, 2-3 swap, 1-2,3-4 swap and other transformations discussed above.

II

1	4	14	15
13	16	2	3
8	5	11	10
12	9	7	6

1	6	12	15
11	16	2	5
8	3	13	10
14	9	7	4

1	7	12	14
10	16	3	5
8	2	13	11
15	9	6	4

III

1	8	12	13
14	11	7	2
15	10	6	3
4	5	9	16

IV

1	4	14	15
16	13	3	2
7	6	12	9
10	11	5	8

VI

1	2	15	16
12	14	3	5
13	7	10	4
8	11	6	9

1	3	14	16
10	13	4	7
15	6	11	2
8	12	5	9

1	3	14	16
12	13	4	5
15	8	9	2
6	10	7	11

1	4	13	16
8	14	3	9
15	5	12	2
10	11	6	7

1	4	13	16
8	15	2	9
14	5	12	3
11	10	7	6

1	4	13	16
12	14	3	5
15	9	8	2
6	7	10	11

1	4	13	16
12	15	2	5
14	9	8	3
7	6	11	10

1	5	12	16
14	11	6	3
15	8	9	2
4	10	7	13

1	6	11	16
7	15	2	10
14	4	13	3
12	9	8	5

1	6	11	16
8	15	2	9
12	3	14	5
13	10	7	4

1	6	11	16
14	12	5	3
15	9	8	2
4	7	10	13

2	1	16	15
11	13	4	6
14	8	9	3
7	12	5	10

2	3	14	15
7	13	4	10
16	6	11	1
9	12	5	8

2	3	14	15
11	13	4	6
16	10	7	1
5	8	9	12

2	5	12	15
13	11	6	4
16	10	7	1
3	8	9	14

3	2	15	14
6	13	4	11
16	7	10	1
9	12	5	8

3	2	15	14
10	13	4	7
16	11	6	1
5	8	9	12

4	1	16	13
6	12	5	11
15	7	10	2
9	14	3	8

5	2	15	12
4	11	6	13
16	7	10	1
9	14	3	8

VIII

1	3	16	14
8	15	2	9
13	6	11	4
12	10	5	7

1	3	16	14
12	15	2	5
13	10	7	4
8	6	9	11

1	4	14	15
9	12	6	7
16	5	11	2
8	13	3	10

1	4	15	14
9	12	7	6
16	5	10	3
8	13	2	11

1	5	16	12
8	14	3	9
10	4	13	7
15	11	2	6

1	5	16	12
10	14	3	7
15	11	6	2
8	4	9	13

1	6	12	15
13	10	8	3
16	7	9	2
4	11	5	14

1	7	16	10
11	13	4	6
14	12	5	3
8	2	9	15

2	4	15	13
9	14	3	8
16	11	6	1
7	5	10	12

2	6	15	11
9	13	4	8
16	12	5	1
7	3	10	14

4	1	14	15
12	9	6	7
13	8	11	2
5	16	3	10

4	1	15	14
8	11	5	10
13	6	12	3
9	16	2	7

4	1	15	14
12	9	7	6
13	8	10	3
5	16	2	11

6	1	12	15
8	10	3	13
11	7	14	2
9	16	5	4

XII

1	2	16	15
13	14	4	3
12	7	9	6
8	11	5	10

1	7	14	12
9	15	4	6
16	10	5	3
8	2	11	13

Henry Ernest Dudeney (1970). Amusements in Mathematics. Dover Publications.
Order 4, Transformations & Patterns. http://www.magic-squares.net/transform.htm